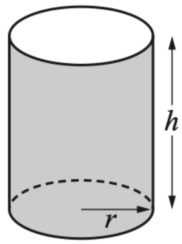
Chapter 12 Differentiations

1.



A container is a circular cylinder, open at one end, with a base radius of *r* cm and a height of *h* cm. The volume of the container is $1000cm^3$. Given that *r* and *h* can vary and that the total outer surface area of the container has a minimum value, find this value.

[8]

$$\pi r^2 h = 1000$$

$$h = \frac{1000}{\pi r^2}$$

$$\text{Surface Area} = \pi r^2 + 2\pi r h$$

$$= \pi r^2 + 2\cancel{\pi}\cancel{r} \times \frac{1000}{\cancel{\pi} r^{\cancel{2}}}$$

$$= \pi r^2 + \frac{2000}{r}$$

$$\frac{dA}{dr} = 2\pi r - \frac{2000}{r^2}$$

$$\frac{dA}{dr} = 0$$

$$2\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{1000}{\pi}$$

$$r = \frac{10}{\sqrt[3]{\pi}} = 6.83$$

$$A = \pi r^2 + \frac{2000}{r} = 439.4\,cm^2$$

2. The radius, r cm, of a circle is increasing at the rate of 5 cms^{-1}. Find, in terms of π, the rate at which the area of the circle is increasing when r = 3.

[4]

$\frac{dr}{dt} = 5$ $\qquad$ $\frac{dA}{dt} = ?$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$= 6\pi$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 6\pi \times 5$

$= 30\pi$

3. The volume, V, of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

The radius, r cm, of a sphere is increasing at the rate of 0.5 cms^{-1}. Find, in terms of π, the rate of change of the volume of the sphere when r = 0.25.

[4]

$V = \frac{4}{3}\pi r^3$ $\qquad$ $\frac{dr}{dt} = 0.5$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi \times (0.25)^2 \times 0.5$

$= \frac{\pi}{8}\ cm^3 s^{-1}$

4. (a) Given that $y = \underbrace{(x^2 - 1)}_{u}\underbrace{\sqrt{5x + 2}}_{v}$, show that $\frac{dy}{dx} = \frac{Ax^2+Bx+C}{2\sqrt{5x+2}}$, where A, B and C are integers.

[5]

$u = x^2 - 1 \qquad u' = 2x$

$v = (5x+2)^{1/2} \qquad v' = \frac{1}{2}(5x+2)^{-1/2} \times 5$

$= \frac{5}{2\sqrt{5x+2}}$

$\frac{dy}{dx} = u'v + v'u$

$= 2x(5x+2)^{1/2} + \frac{5}{2\sqrt{5x+2}} \times (x^2-1)$

$= \frac{4x(5x+2)}{2\sqrt{5x+2}} + \frac{5(x^2-1)}{2\sqrt{5x+2}}$

$= \frac{20x^2 + 8x + 5x^2 - 5}{2\sqrt{5x+2}}$

$= \frac{25x^2 + 8x - 5}{2\sqrt{5x+2}}$

(b) Find the coordinates of the stationary point of the curve $y = (x^2 - 1)\sqrt{5x + 2}$, for $x > 0$. Give each coordinate correct to 2 significant figures.

[3]

$$\frac{dy}{dx} = \frac{25x^2 + 8x - 5}{2\sqrt{5x+2}}$$

$$25x^2 + 8x - 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 + 500}}{50}$$

$$= \frac{-8 \pm \sqrt{564}}{50}$$

$$= \frac{-8 \pm 2\sqrt{141}}{50}$$

$$= \frac{-4 + \sqrt{141}}{25} \quad \text{or} \quad \frac{-4 - \sqrt{141}}{25}$$

$$= 0.31 \quad \text{or} \quad -0.63$$

(reject)

$y = -1.70$

(c) Determine the nature of this stationary point.

[2]

$$\frac{dy}{dx} = \frac{25x^2 + 8x - 5}{2\sqrt{5x+2}} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$u = 25x^2 + 8x - 5$

$u' = 50x + 8$

$v = 2\sqrt{5x+2}$

$v' = (5x+2)^{-1/2} \times 5$

$$\frac{d^2y}{dx^2} = \frac{u'v - v'u}{v^2}$$

$$= \frac{(50x+8)(2\sqrt{5x+2}) - 5(5x+2)^{-1/2}(25x^2 + 8x - 5)}{4(5x+2)}$$

$x = 0.31$

$= 6.26 > 0$

$\therefore$ minimum point

5. (a) Given that $y = \underbrace{x}_{u}\underbrace{\sqrt{x + 2}}_{v}$, show that $\frac{dy}{dx} = \frac{Ax+B}{2\sqrt{x+2}}$, where A and B are constants.

[5]

$u = x$, $u' = 1$

$v = \sqrt{x+2}$, $v' = \frac{1}{2}(x+2)^{-1/2}$

$y' = u'v + v'u$

$= \sqrt{x+2} + \frac{1}{2}(x+2)^{-1/2}(x)$

$= \sqrt{x+2} + \frac{x}{2\sqrt{x+2}}$

$= \frac{2x+4+x}{2\sqrt{x+2}} = \frac{3x+4}{2\sqrt{x+2}}$

(b) Find the exact coordinates of the stationary point of the curve $y = x\sqrt{x + 2}$. [3]

$$y' = \frac{3x+4}{2\sqrt{x+2}}$$

$$\frac{3x+4}{2\sqrt{x+4}} = 0$$

$$3x + 4 = 0$$

$$3x = -4$$

$$x = -\frac{4}{3}$$

$$y = -\frac{4}{3}\sqrt{-\frac{4}{3} + 2} \quad = \frac{-4\sqrt{6}}{9}$$

$$\left(-\frac{4}{3}, -\frac{4\sqrt{6}}{9}\right)$$

(c) Determine the nature of this stationary point. [2]

$$y' = \frac{3x+4 \rightarrow u}{2\sqrt{x+2} \rightarrow v}$$

$$u = 3x+4 \qquad v = 2(x+2)^{\frac{1}{2}}$$

$$u' = 3 \qquad v' = (x+2)^{-\frac{1}{2}}$$

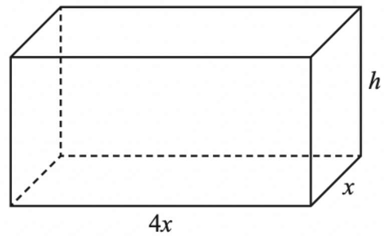
$$y'' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3(2(x+2)^{\frac{1}{2}}) - (x+2)^{-\frac{1}{2}}(3x+4)}{4(x+2)}$$

$x = -\frac{4}{3}$,

$y'' = 1.84 > 0$ $\therefore$ minimum point.

6. In this question all lengths are in centimetres.



The diagram shows a solid cuboid with height *h* and a rectangular base measuring 4*x* by *x*. The volume of the cuboid is 40 cm^3. Given that *x* and *h* can vary and that the surface area of the cuboid has a minimum value, find this value.

[5]

$V = 4x \times x \times h$

$40 = 4x^2h$

$10 = x^2h$

$h = \frac{10}{x^2}$

Surface Area $= 8x^2 + 2xh + 8xh$

$= 8x^2 + 10xh$

$= 8x^2 + \frac{100}{x}$

$\frac{dA}{dx} = 16x - \frac{100}{x^2}$

$16x - \frac{100}{x^2} = 0$

$16x = \frac{100}{x^2}$

$16x^3 = 100$

$x^3 = \frac{25}{4}$

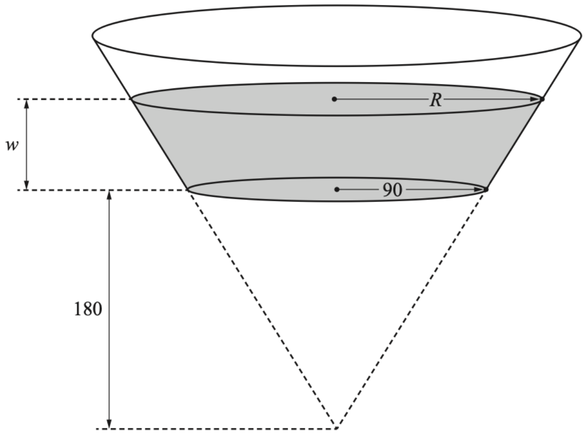
$x = 1.84$

$A = 8x^2 + \frac{100}{x}$

$= 81.4 \text{ cm}^2$

7. In this question all lengths are in centimetres.

The volume, V, of a cone of height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.



The diagram shows a large hollow cone from which a smaller cone of height 180 and base radius 90 has been removed. The remainder has been fitted with a circular base of radius 90 to form a container for water. The depth of water in the container is w and the surface of the water is a circle of radius R.

(a) Find an expression for R in terms of w and show that the volume V of the water in the container is given by $V = \frac{\pi}{12}(w + 180)^3 - 486000\pi$. [3]

$$\text{Vol of water} = \frac{1}{3}\pi R^2(w+180) - \frac{1}{3}\pi 90^2(180)$$
$$= \frac{1}{3}\pi R^2(w+180) - 486000\pi$$

$$\frac{90}{R} = \frac{180}{w+180}$$
$$R = \frac{w+180}{2}$$

$$\text{Vol of water} = \frac{1}{3}\pi\left(\frac{w+180}{2}\right)^2(w+180) - 486000\pi$$
$$= \frac{1}{3}\pi\frac{(w+180)^2}{4} \times (w+180) - 486000\pi$$
$$= \frac{\pi}{12}(w+180)^3 - 486000\pi \text{ (shown)}$$

(b) Water is poured into the container at a rate of $10000 cm^3 s^{-1}$. Find the rate at which the depth of the water is increasing when w = 10.

$\frac{dV}{dt} = 10000 \text{ cm}^3\text{s}^{-1}$ $\quad w = 10$

[4]

$\frac{dw}{dt} = ?$

$V = \frac{\pi}{12}(w+180)^3 - 486000\pi$

$\frac{dV}{dw} = \frac{\pi}{4}(w+180)^2$

$= \frac{\pi}{4} \times (190)^2$

$= 9025\pi$

$\frac{dw}{dt} = \frac{dw}{dV} \times \frac{dV}{dt}$

$= \frac{1}{9025\pi} \times 10000$

$= 0.353$

8. A curve has equation $y = (2x - 1)\sqrt{4x + 3}$.

(a) Show that $\frac{dy}{dx} = \frac{4(Ax+B)}{\sqrt{4x+3}}$ where A and B are constants. [5]

$u = 2x-1$

$u' = 2$

$v = (4x+3)^{1/2}$

$v' = \frac{1}{2}(4x+3)^{-1/2} \times 4$

$= 2(4x+3)^{-1/2}$

$\frac{dy}{dx} = u'v + v'u$

$= 2(4x+3)^{1/2} + 2(4x+3)^{-1/2} \times (2x-1)$

$= 2\sqrt{4x+3} + \frac{2(2x-1)}{\sqrt{4x+3}}$

$= \frac{2(4x+3) + 4x-2}{\sqrt{4x+3}}$

$= \frac{8x+6+4x-2}{\sqrt{4x+3}} = \frac{12x+4}{\sqrt{4x+3}} = \frac{4(3x+1)}{\sqrt{4x+3}}$

(b) Hence write down the x-coordinate of the stationary point of the curve. [1]

$\frac{dy}{dx} = \frac{4(3x+1)}{\sqrt{4x+3}}$

$4(3x+1) = 0$

$3x+1 = 0$

$3x = -1$

$x = -\frac{1}{3}$

(c) Determine the nature of this stationary point. [2]

$\frac{dy}{dx} = \frac{12x+4}{\sqrt{4x+3}}$

$u = 12x+4$

$u' = 12$

$v = (4x+3)^{1/2}$

$v' = \frac{1}{2}(4x+3)^{-1/2} \times 4$

$= 2(4x+3)^{-1/2}$

$\frac{d^2y}{dx^2} = \frac{u'v - v'u}{v^2}$

$= \frac{12(4x+3)^{1/2} - 2(4x+3)^{-1/2}(12x+4)}{4x+3}$

$x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = 9.30 > 0$ $\therefore$ minimum point

9. The equation of a curve is $y = x\sqrt{16 - x^2}$ for $0 \leq x \leq 4$. Find the exact coordinates of the stationary point of the curve. [6]

$u = x \qquad v = (16 - x^2)^{1/2}$

$u' = 1 \qquad v' = \frac{1}{\cancel{2}} \times -\cancel{2}x\,(16 - x^2)^{-1/2}$

$= -x\,(16 - x^2)^{-1/2}$

$\frac{dy}{dx} = u'v + v'u$

$= (16 - x^2)^{1/2} - x(16 - x^2)^{-1/2}\,(x)$

$= (16 - x^2)^{1/2} - x^2\,(16 - x^2)^{-1/2}$

$= \dfrac{16 - x^2 - x^2}{(16 - x^2)^{1/2}} = \dfrac{-2x^2 + 16}{(16 - x^2)^{1/2}}$

$\dfrac{-2x^2 + 16}{(16 - x^2)^{1/2}} = 0$

$-2x^2 + 16 = 0$

$-2x^2 = -16$

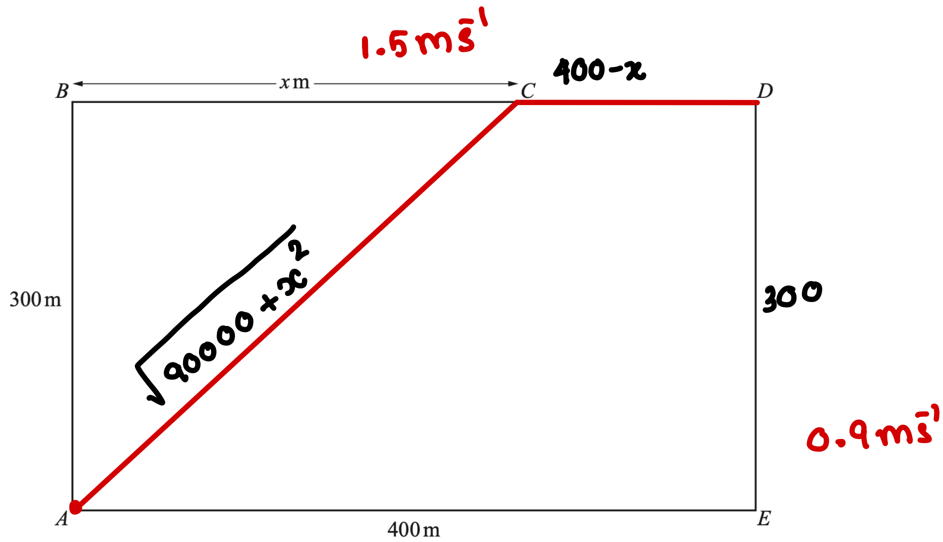
$x^2 = 8$

$x = 2\sqrt{2}$

$y = x\sqrt{16 - x^2}$

$= 8 \qquad \therefore (2\sqrt{2}, 8)$

10.



The rectangle *ABCDE* represents a ploughed field where *AB* = 300 m and *AE* = 400 m . Joseph needs to walk from *A* to *D* in the least possible time. He can walk at 0.9 ms^{-1} on the ploughed field and at 1.5 ms^{-1} on any part of the path *BCD* along the edge of the field. He walks from *A* to *C* and then from *C* to *D*. The distance *BC* = x m.

a. Find, in terms of *x*, the total time, *T* s, Joseph takes for the journey. [3]

$$AC^2 = (300)^2 + x^2$$
$$= 90000 + x^2$$
$$AC = \sqrt{90000 + x^2}$$

$$t = \frac{d}{s}$$

$$\text{total time} = \frac{\sqrt{90000+x^2}}{0.9} + \frac{400-x}{1.5}$$

b. Given that x can vary, find the value of x for which T is a minimum and hence find the minimum value of T.

[6]

$$T = \frac{\sqrt{90000+x^2}}{0.9} + \frac{400-x}{1.5}$$

$$\frac{dT}{dx} = \frac{1}{\cancel{2}\times 0.9} \times (90000+x^2)^{-\frac{1}{2}} \times \cancel{2}x - \frac{1}{1.5}$$

$$0 = \frac{1}{0.9} \times x\,(90000+x^2)^{-\frac{1}{2}} - \frac{1}{1.5}$$

$$\frac{1}{1.5} = \frac{x}{0.9}(90000+x^2)^{-\frac{1}{2}}$$

$$\frac{3}{5} = x\,(90000+x^2)^{-\frac{1}{2}}$$

$$(90000+x^2)^{\frac{1}{2}} = \frac{5x}{3}$$

$$90000+x^2 = \frac{25}{9}x^2$$

$$90000 = \frac{16}{9}x^2$$

$$x^2 = 50625$$

$$x = 225$$

$$T = \frac{\sqrt{90000+225^2}}{0.9} + \frac{400-225}{1.5}$$

$$= 533.\dot{3}$$